Reissner-Nordström metric

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In this note we give derivation of the *Reissner-Nordström metric*. For that we solve the coupled *Einstein-Maxwell equations* for a non-rotating charged spherical black hole. It is assumed that the charge is static, i.e. the current throughout the black hole is uniformly zero and therefore there is no magnetic field produced by the electric charges. However, the existence of the *magnetic monopoles* is assumed and their magnetic field is taken into account.

1. INTRODUCTION

In General Relativity one of famous static solutions to the *Einstein's field equations* is the *Reissner-Nordström metric* describing the geometry of the spacetime surrounding a non-rotating charged spherical black hole. In reality, a highly charged black hole would be quickly neutralized by interactions with matter in its vicinity and therefore such solution is not extremely relevant to realistic astrophysical situations. Nevertheless, charged black holes illustrate a number of important features of more general situations [1]. In almost all General Relativity books known to me (some are [1–6]) authors always leave out the derivation of this metric for the reader as an exercise or tend to obtain it from the more generic *Kerr-Newman metric*. In this note we give detailed derivation of the Reissner-Nordström metric assuming existence of the magnetic monopoles along with electrical charge. To this end, we shall need to solve the coupled *Einstein-Maxwell equations*. We will follow the notation convention used by Sean Caroll in ref. [1]. Because of the spherical symmetry, the *Birkhoff's theorem* suggests the following generic form for the metric in $4D$ spherical coordinates $\{t, r, \theta, \phi\}$ [1]:

$$ds^2 = -e^{2\alpha(r,t)}dt^2 + e^{2\beta(r,t)}dr^2 + r^2 d\Omega^2, \tag{1}$$

where $d\Omega^2$ is the metric on a unit two-sphere,

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2. \tag{2}$$

The Einstein's equation for general relativity is

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{3}$$

where $R_{\mu\nu}$ is the *Ricci tensor* obtained from the *Riemann tensor*,

$$R^{\alpha}{}_{\mu\lambda\nu} = \partial_\lambda \Gamma^\alpha_{\mu\nu} - \partial_\nu \Gamma^\alpha_{\mu\lambda} + \Gamma^\alpha_{\lambda\rho}\Gamma^\rho_{\mu\nu} - \Gamma^\alpha_{\nu\rho}\Gamma^\rho_{\lambda\mu}, \tag{4}$$

by contracting λ with α, R is the *Ricci scalar* $R = g^{\mu\nu}R_{\mu\nu}$, and $T_{\mu\nu}$ is the energy-momentum tensor which in our problem is the one for electromagnetism

$$T_{\mu\nu} = F_{\mu\rho}F_\nu{}^\rho - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}, \tag{5}$$

where $F_{\mu\nu}$ is the *electromagnetic field strength tensor*. Note that $T_{\mu\nu}$ has zero trace,

$$T = g^{\mu\nu}T_{\mu\nu} = g^{\mu\nu}F_{\mu\rho}F_\nu{}^\rho - \frac{1}{4}g^{\mu\nu}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} = 0, \tag{6}$$

since in 4-dimensions $g^{\mu\nu}g_{\mu\nu} = 4$. In Eq. (4) Γ's are the *connection coefficients* given by

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2}g^{\sigma\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}), \tag{7}$$

which holds provided the metric is *compatible* ($\nabla_\mu g^{\mu\nu} = 0$) and connection is *torsion-free* ($\Gamma^\sigma_{\mu\nu} = \Gamma^\sigma_{\nu\mu}$).

Eq. (6) allows us to rewrite the Einstein's equation in the following form

$$R_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{8}$$

Finally, the *Maxwell's equations* are

$$g^{\mu\nu}\nabla_\mu F_{\nu\sigma} = 0, \tag{9}$$

$$\nabla_{[\mu}F_{\nu\rho]} = 0, \tag{10}$$

where ∇ is the covariant derivative operator. The covariant derivative of a rank two tensor $T^{\nu\sigma}$ is defined to be

$$\nabla_\mu T^{\nu\sigma} = \partial_\mu T^{\nu\sigma} + \Gamma^\sigma_{\mu\lambda}T^{\lambda\nu} + \Gamma^\nu_{\mu\lambda}T^{\sigma\lambda}. \tag{11}$$

2. COMPONENTS OF THE ELECTROMAGNETIC FIELD STRENGTH TENSOR

Since there is spherical symmetry, the only non-zero components of the magnetic and electric fields are the radial components which should be independent of θ and

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ϕ. Therefore the radial component of the electric field has a form of

$$E_r = F_{tr} = -F_{rt} = f(r,t). \tag{12}$$

The magnetic field needs more care. First we need to introduce a completely antisymmetric *Levi-Civita symbol*, defined as

$$\tilde{\epsilon}_{\mu\nu\sigma\rho} = \begin{cases} +1, & \text{if } \mu\nu\sigma\rho \text{ is an even permutation of 0123;} \\ -1, & \text{if } \mu\nu\sigma\rho \text{ is an odd permutation of 0123;} \\ 0, & \text{otherwise.} \end{cases} \tag{13}$$

Because $\tilde{\epsilon}_{\mu\nu\sigma\rho}$ is not a tensor - rather a tensor density - the following is defined to be the *Levi-Civita tensor*:

$$\epsilon_{\mu\nu\sigma\rho} = \sqrt{|g|}\tilde{\epsilon}_{\mu\nu\sigma\rho}, \tag{14}$$

where $g = \det g_{\mu\nu}$. Note that using metric $g_{\mu\nu}$ we can lower or raise indices of $\epsilon_{\mu\nu\sigma\rho}$ not of $\tilde{\epsilon}_{\mu\nu\sigma\rho}$. Now we go back to find the magnetic field. The radial component of the magnetic field is given by

$$B_r = g_{11}\epsilon^{01\mu\nu}F_{\mu\nu} = \frac{g_{11}}{\sqrt{|g|}}\tilde{\epsilon}^{01\mu\nu}F_{\mu\nu}$$

$$= \frac{g_{11}}{\sqrt{|g|}}(\tilde{\epsilon}^{0123}F_{23} + \tilde{\epsilon}^{0132}F_{32}) = \frac{2g_{11}}{\sqrt{|g|}}F_{\theta\phi}. \tag{15}$$

From Eq. (1) we see that $g_{11} = g_{rr}(r,t)$ and $|g| \propto r^4 \sin^2\theta$ and since B_r doesn't have angular dependence, $F_{\theta\phi}$ must have the following form

$$F_{\theta\phi} = -F_{\phi\theta} = g(r,t)r^2 \sin\theta. \tag{16}$$

All the remaining components of the electromagnetic field strength tensor are either zero or related to these two through symmetries. Therefore for the electromagnetic field strength tensor we obtain

$$F_{\mu\nu} = \begin{pmatrix} 0 & f(r,t) & 0 & 0 \\ -f(r,t) & 0 & 0 & 0 \\ 0 & 0 & 0 & g(r,t)r^2\sin\theta \\ 0 & 0 & -g(r,t)r^2\sin\theta & 0 \end{pmatrix}. \tag{17}$$

3. COMPONENTS OF THE RICCI TENSOR AND THE STRESS TENSOR

Those of you who have at least once had to solve the Einstein's equation know that the calculation of components of the Ricci tensor is long and tedious work. For this, one should first use Eq. (1) to compute the connection coefficients from Eq. (7), then compute the Riemann tensor in Eq. (4) and eventually contract two of its indices to obtain the Ricci tensor. In this note we won't go through this painful task, instead, take the results that were obtained by Sean Caroll in ref. [1] p.202:

$$R_{tt} = [\partial_t^2\beta + (\partial_t\beta)^2 - \partial_t\alpha\partial_t\beta] + e^{2(\alpha-\beta)}[\partial_r^2\alpha$$

$$+(\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta + \frac{2}{r}\partial_r\alpha],$$

$$R_{rr} = -[\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta - \frac{2}{r}\partial_r\beta]$$

$$+e^{2(\beta-\alpha)}[\partial_t^2\beta + (\partial_t\beta)^2 - \partial_t\alpha\partial_t\beta],$$

$$R_{tr} = \frac{2}{r}\partial_t\beta,$$

$$R_{\theta\theta} = e^{-2\beta}[r(\partial_r\beta - \partial_r\alpha) - 1] + 1,$$

$$R_{\phi\phi} = R_{\theta\theta}\sin^2\theta. \tag{18}$$

For the components of the electromagnetic stress tensor using Eqs. (5) and (17) we obtain

$$\begin{aligned} T_{tt} &= \tfrac{f(r,t)^2}{2}e^{-2\beta(r,t)} + \tfrac{g(r,t)^2}{2}e^{2\alpha(r,t)} \\ T_{rr} &= -\tfrac{f(r,t)^2}{2}e^{-2\alpha(r,t)} - \tfrac{g(r,t)^2}{2}e^{2\beta(r,t)} \\ T_{tr} &= 0 \\ T_{\theta\theta} &= \tfrac{r^2 g(r,t)^2}{2} + \tfrac{r^2 f(r,t)^2}{2}e^{-2(\alpha(r,t)+\beta(r,t))} \\ T_{\phi\phi} &= T_{\theta\theta}\sin^2\theta. \end{aligned} \tag{19}$$

From Eqs. (18) and (19) we have $R_{tr} = 0$ which gives $\beta = \beta(r)$. Using this fact and Eq. (12), we obtain $e^{2\alpha(r,t)}R_{rr} + e^{2\beta(r)}R_{tt} = 0$. Solving this yields $\alpha(r,t) + \beta(r) = const.$ But we can redefine the time coordinate in Eq. (1) by replacing $dt \to e^{const.}dt$ so that

$$\alpha(r,t) = \alpha(r) = -\beta(r). \tag{20}$$

4. SOLVING THE MAXWELL EQUATIONS

Now lets solve the Maxwell equations for the form of the electromagnetic field strength tensor given in Eq. (17). For the r component of the Eq. (9) we have

$$\partial_t F_{tr} - \Gamma^\alpha_{tt}F_{\alpha r} - \Gamma^\alpha_{tr}F_{t\alpha} = 0,$$

or carrying out the summation over α gives

$$\partial_t F_{tr} - F_{tr}(\Gamma^r_{tt} + \Gamma^r_{tr}) = 0.$$

Since the metric is diagonal and β doesn't depend on time, $\Gamma^t_{tt} = 0$ and $\Gamma^r_{tr} = \partial_t\beta = 0$ and from above equation we have $\partial_t F_{tr} = 0$ implying that the tr component of the electromagnetic field strength tensor is not time dependent:

$$F_{tr} = f(r). \tag{21}$$

To find the explicit form of f, we will make use of the following identity: for given any antisymmetric rank two tensor, $T^{\mu\nu}$, and diagonal metric the following identity is true (see Eq. (A3))

$$\nabla_\mu T^{\mu\nu} = \frac{1}{\sqrt{|g|}}\partial_\mu(\sqrt{|g|}T^{\mu\nu}). \tag{22}$$

If we take into account Eq. (20), for our metric we have $\sqrt{|g|} = r^2\sin\theta$. Now if we use metric compatibility condition to raise the indices of the electromagnetic field strength tensor in Eq. (9) and apply the above identity, we obtain

$$\nabla_\mu F^{\mu\nu} = \frac{1}{r^2\sin\theta}\partial_\mu(r^2\sin\theta F^{\mu\nu}) = 0. \tag{23}$$

For the t component we have $\partial_r(r^2F^{rt}) = \partial_r(r^2g^{rr}g^{tt}F_{rt}) = \partial_r(r^2f) = 0$, or

$$f(r) = \frac{const.}{r^2}. \tag{24}$$

For the constant the Gauss's flux theorem gives $const. = Q/\sqrt{4\pi}$ where Q is the total *electric charge* of black hole.

Now lets find $g(r,t)$ which is related to the magnetic filed. For this we need to solve Eq. (10) which in explicit form reads

$$\nabla_\mu F_{\nu\rho} + \nabla_\nu F_{\rho\mu} + \nabla_\rho F_{\mu\nu} = 0. \tag{25}$$

If we expand above equation using Eq. (11), all terms which contain connection coefficients vanish and we are left with the ordinary partial derivatives

$$\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0. \tag{26}$$

Considering $\mu = t$, $\nu = \phi$ and $\rho = \theta$ combination and using the fact that $F_{\theta t} = F_{t\phi} = 0$, we obtain $\partial_t F_{\theta\phi} = 0$ which means that $g(r,t)$ is time independent. Doing the same for $\mu = r$, $\nu = \theta$ and $\rho = \phi$ combination leads to $\partial_r F_{\theta\phi} = 0$ or $\partial_r(r^2g(r)) = 0$. Thus

$$g(r,t) = \frac{const.}{r^2}. \tag{27}$$

Similar to the electric charges, the Gauss's flux theorem for the magnetic field gives $const. = P/\sqrt{4\pi}$ where P is the total *magnetic charge* of black hole. Finally, for the electromagnetic field strength tensor we obtain

$$F_{\mu\nu} = \frac{1}{\sqrt{4\pi}}\begin{pmatrix} 0 & Qr^{-2} & 0 & 0 \\ -Qr^{-2} & 0 & 0 & 0 \\ 0 & 0 & 0 & P\sin\theta \\ 0 & 0 & -P\sin\theta & 0 \end{pmatrix}. \tag{28}$$

5. THE REISSNER-NORDSTRÖM METRIC

Now we are left with only one unknown variable, $\alpha(r)$ which is given in Eq. (20). To this end, one equation is enough to determine the unknown. Lets consider the $\theta\theta$ component of the Einstein's equation, Eq. (7):

$$R_{\theta\theta} = 8\pi T_{\theta\theta}. \tag{29}$$

Substituting $R_{\theta\theta}$ and $T_{\theta\theta}$ from Eqs. (18) and (19) into the above equation and using Eqs. (20) and (28), we obtain

$$\partial_r(re^{2\alpha}) = 1 - \frac{G}{r^2}(Q^2 + P^2),$$

or

$$e^{2\alpha} = 1 + \frac{R_S}{r} + \frac{G}{r^2}(Q^2 + P^2). \tag{30}$$

In the absence of charges, this should reduce to the *Schwarzschild* solution which allows us to take the constant to be $R_S = 2GM$ where M is interpreted as the mass of black hole and G is the *Newton*'s gravitational constant.

Finally, upon substitution of Eqs. (20) and (30) into Eq. (1), the **Reissner-Nordström metric** is readily found:

$$ds^2 = -\triangle dt^2 + \triangle^{-1}dr^2 + r^2d\Omega^2, \tag{31}$$

where

$$\triangle = 1 - \frac{2GM}{r} + \frac{G}{r^2}(Q^2 + P^2). \tag{32}$$

In summery, we have solved the coupled Einstein-Maxwell equations and found the metric which describes the geometry of the spacetime surrounding a non-rotating black hole assuming it has static electric and magnetic charges.

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APPENDIX A: DERIVATION OF EQ. (22)

We start with

$$\Gamma^{\mu}_{\mu\lambda} = \frac{1}{\sqrt{|g|}}\partial_\lambda \sqrt{|g|} \tag{A1}$$

which is readily shown for a diagonal metric. Then taking the above identity for granted, for a rank two tensor $T^{\mu\nu}$ we can write

$$\nabla_\mu T^{\mu\nu} = \partial_\mu T^{\mu\nu} + \Gamma^{\mu}_{\mu\alpha} T^{\alpha\mu} + \Gamma^{\nu}_{\mu\alpha} T^{\mu\alpha}$$

$$= \partial_\mu T^{\mu\nu} + \frac{1}{\sqrt{|g|}}\partial_\alpha(\sqrt{|g|}) T^{\alpha\nu} + \Gamma^{\nu}_{\mu\alpha} T^{\mu\alpha}$$

$$= \frac{1}{\sqrt{|g|}}\partial_\alpha(\sqrt{|g|} T^{\alpha\nu}) + \Gamma^{\nu}_{\mu\alpha} T^{\mu\alpha}. \tag{A2}$$

Since the connection coefficients are torsion free, for an asymmetric tensor $T^{\mu\nu}$ the last term is zero which leads to the following identity:

$$\nabla_\mu T^{\mu\nu} = \frac{1}{\sqrt{|g|}}\partial_\mu(\sqrt{|g|} T^{\mu\nu}). \tag{A3}$$

Q.E.D.