Reissner-Nordström metric

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In this note we give derivation of the Reissner-Nordström metric. For that we solve the coupled Einstein-Maxwell equations for a non-rotating charged spherical black hole. It is assumed that the charge is static, i.e. the current throughout the black hole is uniformly zero and therefore there is no magnetic field produced by the electric charges. However, the existence of the magnetic monopoles is assumed and their magnetic field is taken into account.

1. INTRODUCTION

In General Relativity one of famous static solutions to the Einstein’s field equations is the Reissner-Nordström metric describing the geometry of the spacetime surrounding a non-rotating charged spherical black hole. In reality, a highly charged black hole would be quickly neutralized by interactions with matter in its vicinity and therefore such solution is not extremely relevant to realistic astrophysical situations. Nevertheless, charged black holes illustrate a number of important features of more general situations [1]. In almost all General Relativity books known to me (some are [1–6]) authors always leave out the derivation of this metric for the reader as an exercise or tend to obtain it from the more generic Kerr-Newman metric. In this note we give detailed derivation of the Reissner-Nordström metric assuming existence of the magnetic monopoles along with electrical charge. To this end, we shall need to solve the coupled Einstein-Maxwell equations. We will follow the notation convention used by Sean Caroll in ref. [1]. Because of the spherical symmetry, the Birkhoff’s theorem suggests the following generic form for the metric in 4D spherical coordinates \{t, r, \theta, \phi\} [1]:

\[
ds^2 = -e^{2\alpha(r,t)}dt^2 + e^{2\beta(r,t)}dr^2 + r^2d\Omega^2, \tag{1}
\]

where \(d\Omega^2\) is the metric on a unit two-sphere,

\[
d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2. \tag{2}
\]

The Einstein’s equation for general relativity is

\[
R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{3}
\]

where \(R_{\mu\nu}\) is the Ricci tensor obtained from the Riemann tensor,

\[
R^\alpha^\beta_{\mu\lambda
\nu} = \partial_\lambda \Gamma^\alpha^\beta_{\mu\nu} - \partial_\nu \Gamma^\alpha^\beta_{\mu\lambda} + \Gamma^\alpha^\beta_{\mu\rho} \Gamma^\rho^\gamma^\lambda - \Gamma^\alpha^\gamma^\nu \Gamma^\rho^\lambda^\beta, \tag{4}
\]

by contracting \(\lambda\) with \(\alpha\), \(R\) is the Ricci scalar \(R = g^\mu\nu R_{\mu\nu}\), and \(T_{\mu\nu}\) is the energy-momentum tensor which in our problem is the one for electromagnetism

\[
T_{\mu\nu} = F_{\mu\sigma} F^\nu^\sigma - \frac{1}{4} g_{\mu\nu} F^\rho^\sigma F_{\rho\sigma}, \tag{5}
\]

where \(F_{\mu\nu}\) is the electromagnetic field strength tensor. Note that \(T_{\mu\nu}\) has zero trace,

\[
T = g^{\mu\nu} T_{\mu\nu} = g^{\mu\nu} F_{\mu\rho} F^\nu^\rho - \frac{1}{4} g^{\mu\nu} g_{\rho\sigma} F_{\rho\sigma} F^{\rho\sigma} = 0, \tag{6}
\]

since in 4-dimensions \(g^{\mu\nu} g_{\mu\nu} = 4\). In Eq. (4) \(\Gamma^\alpha^\beta_{\mu\nu}\) are the connection coefficients given by

\[
\Gamma^\sigma^\nu_{\mu\nu} = \frac{1}{2} g^{\sigma\rho} (\partial_\mu g_{\rho\nu} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}), \tag{7}
\]

which holds provided the metric is compatible \((\nabla_\mu g^{\mu\nu} = 0)\) and connection is torsion-free \((\Gamma^\sigma^\nu_{\mu\nu} = \Gamma^\sigma^\nu_{\nu\mu})\).

Eq. (6) allows us to rewrite the Einstein’s equation in the following form

\[
R_{\mu\nu} = 8\pi GT_{\mu\nu}. \tag{8}
\]

Finally, the Maxwell’s equations are

\[
g^{\mu\nu} \nabla_\mu F_{\nu\sigma} = 0, \tag{9}
\]

\[
\nabla_\mu F_{\nu\rho} = 0, \tag{10}
\]

where \(\nabla\) is the covariant derivative operator. The covariant derivative of a rank two tensor \(T^{\nu\sigma}\) is defined to be

\[
\nabla_\mu T^{\nu\sigma} = \partial_\mu T^{\nu\sigma} + \Gamma^\sigma^\nu_{\mu\lambda} T^{\lambda\nu} + \Gamma^\nu^\sigma_{\mu\lambda} T^{\lambda\nu}. \tag{11}
\]

2. COMPONENTS OF THE ELECTROMAGNETIC FIELD STRENGTH TENSOR

Since there is spherical symmetry, the only non-zero components of the magnetic and electric fields are the radial components which should be independent of \(\theta\) and
\[ E_r = F_{rt} = -F_{tt} = f(r, t). \] (12)

The magnetic field needs more care. First we need to introduce a completely antisymmetric Levi-Civita symbol, defined as

\[ \epsilon_{\mu\nu\rho\sigma} = \begin{cases} +1, & \text{if } \mu\nu\rho\sigma \text{ is an even permutation of 0123;} \\ -1, & \text{if } \mu\nu\rho\sigma \text{ is an odd permutation of 0123;} \\ 0, & \text{otherwise}. \end{cases} \] (13)

Because \( \epsilon_{\mu\nu\rho\sigma} \) is not a tensor - rather a tensor density - the following is defined to be the Levi-Civita tensor:

\[ \epsilon_{\mu\nu\rho\sigma} = \sqrt{|g|} \epsilon_{\mu\nu\rho\sigma}, \] (14)

where \( g = \det g_{\mu\nu} \). Note that using metric \( g_{\mu\nu} \) we can lower or raise indices of \( \epsilon_{\mu\nu\rho\sigma} \) not of \( \epsilon_{\mu\nu\rho\sigma} \). Now we go back to find the magnetic field. The radial component of the magnetic field is given by

\[ B_r = g_{11} \epsilon^{01\mu\nu} F_{\mu\nu} = \frac{g_{11}}{|g|} \epsilon^{01\mu\nu} F_{\mu\nu} = \frac{g_{11}}{|g|} (\epsilon^{0123} F_{23} + \epsilon^{0132} F_{32}) = \frac{2g_{11}}{|g|} F_{\theta\phi}. \] (15)

From Eq. (1) we see that \( g_{11} = g_{rr}(r, t) \) and \( |g| \propto r^4 \sin^2 \theta \) and since \( B_r \) doesn't have angular dependence, \( F_{\theta\phi} \) must have the following form

\[ F_{\theta\phi} = -F_{\phi\theta} = g(r, t) r^2 \sin \theta. \] (16)

All the remaining components of the electromagnetic field strength tensor are either zero or related to these two through symmetries. Therefore for the electromagnetic field strength tensor we obtain

\[ F_{\mu\nu} = \begin{pmatrix} 0 & f(r, t) & 0 & 0 \\ -f(r, t) & 0 & 0 & 0 \\ 0 & 0 & 0 & g(r, t) r^2 \sin \theta \\ 0 & 0 & -g(r, t) r^2 \sin \theta & 0 \end{pmatrix}. \] (17)

3. COMPONENTS OF THE RICCI TENSOR AND THE STRESS TENSOR

Those of you who have at least once had to solve the Einstein's equation know that the calculation of components of the Ricci tensor is long and tedious work. For this, one should first use Eq. (1) to compute the connection coefficients from Eq. (7), then compute the Riemann tensor in Eq. (4) and eventually contract two of its indices to obtain the Ricci tensor. In this note we won't go through this painful task, instead, take the results that were obtained by Sean Caroll in ref. [1] p.202:

\[ R_{tt} = [\partial_r^2 \beta + (\partial_t \beta)^2 - \partial_t \alpha \partial_r \beta] + e^{2(\alpha - \beta)} \partial_r^2 \alpha \]  
\[ + (\partial_r \alpha)^2 - \partial_t \alpha \partial_r \beta + \frac{2}{r} \partial_r \alpha], \]

\[ R_{rr} = -[\partial_r^2 \alpha + (\partial_r \alpha)^2 - \partial_t \alpha \partial_r \beta - \frac{2}{r} \partial_r \beta] \]  
\[ + e^{2(\beta - \alpha)} [\partial_t^2 \beta + (\partial_t \beta)^2 - \partial_r \alpha \partial_t \beta], \]

\[ R_{\theta\theta} = -[\partial_r^2 \beta + (\partial_t \beta)^2 - \partial_t \alpha \partial_r \beta - \frac{2}{r} \partial_r \beta] \]  
\[ + e^{2(\beta - \alpha)} [\partial_t^2 \beta + (\partial_t \beta)^2 - \partial_r \alpha \partial_t \beta], \]

\[ R_{\phi\phi} = -[\partial_r^2 \beta + (\partial_t \beta)^2 - \partial_t \alpha \partial_r \beta - \frac{2}{r} \partial_r \beta] \]  
\[ + e^{2(\beta - \alpha)} [\partial_t^2 \beta + (\partial_t \beta)^2 - \partial_r \alpha \partial_t \beta], \]

\[ R_{\theta\phi} = R_{\phi\theta} \sin^2 \theta. \] (18)

For the components of the electromagnetic stress tensor using Eqs. (5) and (17) we obtain

\[ T_{tt} = \frac{f(r, t)}{2} e^{-2\beta(r, t)} + \frac{g(r, t)^2}{2} e^{2\alpha(r, t)} \]  
\[ T_{rr} = -\frac{f(r, t)}{2} e^{-2\alpha(r, t)} - \frac{g(r, t)^2}{2} e^{2\beta(r, t)} \]  
\[ T_{\theta\theta} = \frac{r^2 g(r, t)^2}{2} + \frac{r^2 f(r, t)^2}{2} e^{-2(\alpha(r, t) + \beta(r, t))} \]  
\[ T_{\phi\phi} = \frac{r^2 g(r, t)^2}{2} + \frac{r^2 f(r, t)^2}{2} e^{-2(\alpha(r, t) + \beta(r, t))} \] (19)

From Eqs. (18) and (19) we have \( R_{rr} = 0 \) which gives \( \beta = \beta(r) \). Using this fact and Eq. (12), we obtain \( e^{2\alpha(r, t)} R_{\theta\theta} + e^{2\beta(r)} R_{tt} = 0 \). Solving this yields \( \alpha(r, t) + \beta(r) = const \). But we can redefine the time coordinate in Eq. (1) by replacing \( dt \rightarrow e^{const} \sin \theta \) so that

\[ \alpha(r, t) = \alpha(r) = -\beta(r). \] (20)

4. SOLVING THE MAXWELL EQUATIONS

Now lets solve the Maxwell equations for the form of the electromagnetic field strength tensor given in Eq. (17). For the \( r \) component of the Eq. (9) we have

\[ \partial_t F_{tr} - \Gamma^t_{tt} F_{tr} - \Gamma^r_{tt} F_{ta} = 0, \]

or carrying out the summation over \( \alpha \) gives

\[ \partial_t F_{tr} - F_{tr}(\Gamma^t_{tt} + \Gamma^r_{tt}) = 0. \]

Since the metric is diagonal and \( \beta \) doesn't depend on time, \( \Gamma^t_{tt} = 0 \) and \( \Gamma^r_{tt} = \partial_t \beta = 0 \) and from above equation we have \( \partial_t F_{tr} = 0 \) implying that the \( tr \) component of the electromagnetic field strength tensor is not time dependent:

\[ F_{tr} = f(r). \] (21)

To find the explicit form of \( f \), we will make use of the following identity: for given any antisymmetric rank two
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Now we are left with only one unknown variable, $\alpha(r)$ which is given in Eq. (20). To this end, one equation is enough to determine the unknown. Let us consider the $\theta\theta$ component of the Einstein’s equation, Eq. (7):

$$R_{\theta\theta} = 8\pi T_{\theta\theta}. \quad (29)$$

Substituting $R_{\theta\theta}$ and $T_{\theta\theta}$ from Eqs. (18) and (19) into the above equation and using Eqs. (20) and (28), we obtain

$$\partial_r (r e^{2\alpha}) = 1 - \frac{G}{r^2} (Q^2 + P^2),$$

or

$$e^{2\alpha} = 1 + \frac{R_S}{r} + \frac{G}{r^2} (Q^2 + P^2). \quad (30)$$

In the absence of charges, this should reduce to the Schwarzschild solution which allows us to take the constant to be $R_S = 2GM$ where $M$ is interpreted as the mass of black hole and $G$ is the Newton’s gravitational constant.

Finally, upon substitution of Eqs. (20) and (30) into Eq. (1), the Reissner-Nordström metric is readily found:

$$ds^2 = -\Delta dt^2 + \frac{1}{\Delta} dr^2 + r^2 d\Omega^2, \quad (31)$$

where

$$\Delta = 1 - \frac{2GM}{r} + \frac{G}{r^2} (Q^2 + P^2). \quad (32)$$

In summary, we have solved the coupled Einstein-Maxwell equations and found the metric which describes the geometry of the spacetime surrounding a non-rotating black hole assuming it has static electric and magnetic charges.

Appendix A: Derivation of Eq. (22)

We start with

$$\Gamma^\mu_{\mu\lambda} = \frac{1}{\sqrt{|g|}} \partial_\lambda \sqrt{|g|}$$

(A1)

which is readily shown for a diagonal metric. Then taking the above identity for granted, for a rank two tensor $T^\mu_\nu$ we can write

$$\nabla_\mu T^\mu_\nu = \partial_\mu T^\mu_\nu + \Gamma^\mu_\mu T^\alpha_\mu + \Gamma^\nu_\mu T^\mu_\alpha$$

$$= \partial_\mu T^\mu_\nu + \frac{1}{\sqrt{|g|}} \partial_\alpha (\sqrt{|g|}) T^\alpha_\nu + \Gamma^\nu_\mu T^\mu_\alpha$$

$$= \frac{1}{\sqrt{|g|}} \partial_\alpha (\sqrt{|g|}) T^\alpha_\nu + \Gamma^\nu_\mu T^\mu_\alpha.$$  

(A2)

Since the connection coefficients are torsion free, for an asymmetric tensor $T^\mu_\nu$ the last term is zero which leads to the following identity:

$$\nabla_\mu T^\mu_\nu = \frac{1}{\sqrt{|g|}} \partial_\mu (\sqrt{|g|}) T^\mu_\nu.$$  

(A3)

Q.E.D.