Problem 1

Let \( *F_{\mu\nu} \) be the field dual to \( F_{\mu\nu} \) with

\[
*F_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\lambda\rho} F^{\lambda\rho}
\]

Calculate \( *F_{ij} \) and \( *F_{4i} = i *F_{0i} \) in terms of \( E_i \) and \( B_i \). What is \( *F_{\mu\nu} F^{\mu\nu} \) in terms of \( E \) and \( B \)?

Solution:

To solve the problem we will use the metric signature \((+---)\) and \( \epsilon_{0123} = -1 \).

In this convention the field-strength tensor is

\[
F^{\mu\nu} = \begin{pmatrix}
0 & -E_x & -E_y & -E_z \\
E_x & 0 & -B_z & B_y \\
E_y & B_z & 0 & -B_x \\
E_z & -B_y & B_x & 0
\end{pmatrix}
\]

(2)

Where the index \( \mu = 0, 1, 2, 3 \) or \( (t,x,y,z) \) labels the rows, and the index \( \nu \) the columns.

Using (1), (2) and antisymmetric property of both \( F_{\mu\nu} \) and \( \epsilon_{\mu\nu\lambda\rho} \) we can easily compute components of \( *F_{\mu\nu} \).

\[
*F_{12} = \frac{1}{2} (\epsilon_{1230} F^{30} + \epsilon_{1203} F^{03}) = \epsilon_{1230} F^{30} = E_z
\]

\[
*F_{13} = \frac{1}{2} (\epsilon_{1320} F^{20} + \epsilon_{1302} F^{02}) = -\epsilon_{1320} F^{20} = -E_y
\]
\[ *F_{23} = \frac{1}{2}(\epsilon_{2310} F^{10} + \epsilon_{2301} F^{01}) = -\epsilon_{3210} F^{10} = E_x \]

\[ *F_{01} = \frac{1}{2}(\epsilon_{0123} F^{23} + \epsilon_{0132} F^{32}) = \epsilon_{0123} F^{23} = B_x \]

\[ *F_{02} = \frac{1}{2}(\epsilon_{0213} F^{13} + \epsilon_{0231} F^{31}) = -\epsilon_{0123} F^{13} = B_y \]

\[ *F_{03} = \frac{1}{2}(\epsilon_{0312} F^{12} + \epsilon_{0321} F^{21}) = -\epsilon_{3012} F^{12} = B_z \]

Therefore the dual field-strength tensor is

\[ *F_{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & E_z & -E_y \\ -B_y & -E_z & 0 & E_x \\ -B_z & E_y & -E_x & 0 \end{pmatrix} \] (3)

And if \( i < j \)

\[ *F_{\mu\nu} F^{\mu\nu} = 2(*F_{0l} F^{0l} + *F_{ij} F^{ij}) = -2(B_x E_x + B_y E_y + B_z E_z + B_x E_x + B_y E_y + B_z E_z) = -4E \cdot B \]

Note that the left hand side of (4) being tensorial expression is coordinate independent, and so is the right hand side. As it can be seen from the right hand side of (4), \( *F_{\mu\nu} F^{\mu\nu} \) is a pseudoscalar (changes sign under a parity inversion) quantity. \( *F_{\mu\nu} F^{\mu\nu} \) is also one of two (another \( F_{\mu\nu} F^{\mu\nu} \)) fundamental invariants of the electromagnetic field.

**Problem 2**

If there are elementary magnetic monopoles, then there will be a magnetic current, \( J_{M\mu} \), to which \( \partial_{\lambda} *F_{\lambda\mu} \) is proportional. Find this equation for \( \partial_{\lambda} *F_{\lambda\mu} \) using its similarity to \( \partial_{\lambda} F_{\lambda\mu} \) in the presence of an electric current. Prove from this equation that \( \partial_{\mu} J_{M\mu} = 0 \).

**Solution:**

Using (2) we can write

\[ F_{\lambda\mu} = \eta_{\lambda\rho} F^{\sigma\rho} \eta_{\rho\mu} \]

\[ = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \]
\[
\begin{pmatrix}
0 & E_x & E_y & E_z \\
-E_x & 0 & B_z & -B_y \\
-E_y & -B_z & 0 & B_x \\
-E_z & B_y & -B_x & 0
\end{pmatrix}
\]
which satisfies
\[
\partial_\lambda F_{\lambda\mu} = \frac{4\pi}{c} J_\mu. \quad (5)
\]
On the other hand we have already computed \( *F_{\lambda\mu} \) which is given by
\[
*F_{\lambda\mu} = \begin{pmatrix}
0 & B_x & B_y & B_z \\
-B_x & 0 & E_z & -E_y \\
-B_y & -E_z & 0 & E_x \\
-B_z & E_y & -E_x & 0
\end{pmatrix}. \quad (6)
\]
If we compare \( *F_{\lambda\mu} \) to \( F_{\lambda\mu} \), we see that they are related by \( \mathbf{B} \rightarrow \mathbf{E} \). Thus we expect that if there are magnetic monopoles in the nature, \( *F_{\lambda\mu} \) will satisfy the same equation as one for \( F_{\lambda\mu} \),
\[
\partial_\lambda *F_{\lambda\mu} = \frac{4\pi}{c} J_{M\mu} \quad (7)
\]
where \( J_{M\mu} \) is a magnetic current.

For the second part of the problem we use antisymmetry of \( *F_{\lambda\mu} \) and the symmetric property of partial derivatives:
\[
\partial_\mu \partial_\lambda *F_{\lambda\mu} = \partial_\lambda \partial_\mu *F_{\mu\lambda} = -\partial_\lambda \partial_\mu *F_{\lambda\mu} = -\partial_\mu \partial_\lambda *F_{\lambda\mu}
\]
If we put left and right hand sides together, we get
\[
2\partial_\mu \partial_\lambda *F_{\lambda\mu} = 0
\]
which implies
\[
\partial_\mu J_{M\mu} = 0.
\]