Problem 1

A magnetic monopole is located at the origin so that \( B = \frac{g}{4\pi r^3} \) and

\[
\int_{S^2} B \cdot dS = g
\]  

(1)
on a sphere \( S^2 \) enclosing the origin. Show that there is no singularity free vector potential \( A \), such that \( B = \nabla \times A \) using Stokes’ theorem. Thus \( A \) is singular at least at one point on every two sphere enclosing origin, or it is singular at least along a line (the Dirac String) from origin to infinity.

Solution:

Using Stokes’ theorem we can write

\[
\int_{S^2} B \cdot dS = \int_{S^2} \nabla \times A \cdot dS = \int_{\partial S} A \cdot dL
\]  

(2)

where \( \partial S \) is a positive oriented, simple, closed, piecewise-smooth boundary of the \( S^2 \). Since two sphere is a closed surface, it doesn’t have a boundary and the requirement that \( A \) is smooth on the \( S^2 \) implies that the right hand side of eq. (2) must vanish which contradicts eq. (1). This means \( A \) must have singularity at least at one point on \( S^2 \), or in other words \( S^2 \) is not closed surface, i.e. it has a hole which is bounded by \( \partial S \). Therefore every sphere enclosing origin, has at least one topological defect, hole. Thus there exist continuous singularity line starting from the origin and passes through these holes to infinity. This line is what called the Dirac String.